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1a $f_n(x)$ uniform naar 0

$\Rightarrow$

voor elke $\varepsilon > 0$ $\exists\, N = N_\varepsilon \in \mathbb{N}$ zdd.

$\| f_n(x) - 0 \| \leq \varepsilon$ $\forall x \in [0,1]$ als $n \geq N_\varepsilon$

$\| f_n(x) \| \leq \varepsilon$ $\forall x \in [0,1]$ als $n \geq N_\varepsilon$

laten we $n \to \infty$, dan hebben we zeker $n \geq N_\varepsilon$,
~~daarvoor~~ en we hebben $\frac{1}{n} \in [0,1]$

$\Rightarrow \lim_{n\to\infty} f_n(1/n) = 0$

b $f_n(x) = nx(1-x)^n$

$f_n(1/n) = n\frac{1}{n}(1-\frac{1}{n})^n = (1-\frac{1}{n})^n$

$\lim_{n\to\infty} f_n(1/n) = \lim_{n\to\infty} (1-\frac{1}{n})^n = 1/e \neq 0$

we weten nu dat

1) $f_n : [0,1] \to \mathbb{C}$ convergeert uniform naar 0

$\Rightarrow \lim_{n\to\infty} f_n(1/n) = 0$ (uit a)

2) $\lim_{n\to\infty} f_n(1/n) \neq 0$

hieruit volgt: f_n convergeert niet uniform naar 0 op $[0,1]$

c $x \in [a,1]$ met $0 < a < 1$

$\Rightarrow$

$0 < a \leq x \leq 1$ met $a < 1$

$\Rightarrow$

$0 \leq 1-x \leq 1-a < 1$ en $x \neq 0$

we hebben nu $f_n(x) = nx(1-x)^n$ op $[a,1]$, $0 < a < 1$

$= x \cdot nb^n$ met $x \neq 0$ en $0 \leq b < 1$

$(b = 1-x)$

dan weten we : ...

⇒ $f_n(x)$ uniform conv. op $[a,1]$ $0<a<1$

2a $f \in L^1(0,\infty)$ $\quad f_n(x) = e^{-nx} f(x)$ $\quad \leq |f(x)|$

$=0$ voor $x \neq 0$

$$F(x) = \lim_{n\to\infty}\left(e^{-nx} f(x)\right) \neq f(x)$$

$= f(0)$ als $x=0$

-4

neem nu $g(x) = f_n(x)\cdot e^{nx}$

$$= e^{-nx}\cdot f(x) \cdot e^{nx} = f(x)$$

hoezo? je weet niks van f

dan geldt $g(x) = f(x) \geq 0 \quad \forall x$

en $\int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}} f(x)\,dx < \infty$

$(f \in L^1(0,\infty))$

betekent $\int |f(x)|\,dx < \infty$

verder: $\quad f_n(x) = e^{-nx} f(x)$

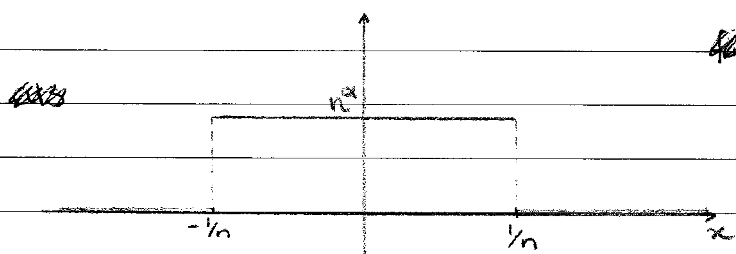
$1 \leq e^{nx}$ voor $x \in [0,\infty)$!

$$e^{nx} f_n(x) = f(x)$$

$$|f_n(x)| \leq f(x) \quad \text{b.o.}$$

$$\Rightarrow \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx$$

b $\quad f_n(x) = \begin{cases} n^\alpha & \text{als } |x| \leq 1/n \\ 0 & \text{anders} \end{cases}$



$0 \leq f_n(x) \quad \forall x$

nu geldt

dit mag niet zomaar

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n(x)\right) dx$$

$$= \int_{-\frac{1}{n}}^{\frac{1}{n}} \lim_{n\to\infty} n^\alpha \,dx$$

$$= \frac{2\lim_{n\to\infty} n^\alpha}{n} = 2\lim_{n\to\infty} n^{\alpha-1}$$

als $0 \leq f_n(x) \leq f_{n+1}(x)$ $\quad$ tussen $\frac{1}{n}$ en $\frac{1}{n+1}$ is

$0 \leq n^\alpha \leq (n+1)^\alpha$ $\quad$ $f_n(x) > f_{n+1}(x) = 0$

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dus voor $\alpha \geq 0$ geldt ?

is algemeen → $\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx = 2\lim_{n\to\infty} n^{\alpha-1}$

3 $$= \begin{cases} 0 & \alpha < 1 \\ \infty & \alpha \geq 1 \end{cases}$$

dus $\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx = \infty$ voor $\alpha \geq 1$

er vindt hier monotone convergentie plaats

3a $f:[-\pi,\pi]\to\mathbb{C}$ 2π-periodiek : $f(t)=te^{it}$

$$c_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-it}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{it}e^{-it}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt$$

$$= \frac{1}{2\pi}\left[\tfrac{1}{2}t^2\right]_{-\pi}^{\pi} = \frac{\pi^2-(-\pi)^2}{4\pi} = 0$$

$$\underset{\substack{n\neq 1\\ 1-n\neq 0}}{c_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{it}e^{-int}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{it(1-n)}\,dt$$

$$= \frac{1}{2\pi}\left(\frac{te^{it(1-n)}}{i(1-n)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{it(1-n)}}{i(1-n)}\,dt \right)$$

$$= \frac{1}{2\pi}\left(\frac{\pi e^{i\pi(1-n)} + \pi e^{-i\pi(1-n)}}{i(1-n)} + \left[\frac{e^{it(1-n)}}{(1-n)^2}\right]_{-\pi}^{\pi} \right)$$

$$= \frac{1}{2\pi}\left(\frac{2\pi\cos(\pi(1-n))}{i(1-n)} + \frac{e^{i\pi(1-n)} - e^{-i\pi(1-n)}}{(1-n)^2} \right)$$

$$= \frac{\cos(\pi(1-n))}{i(1-n)} + i\,\frac{e^{i\pi(1-n)} - e^{-i\pi(1-n)}}{2\pi i(1-n)^2}$$

$$= \frac{(-1)^{1-n}}{i(1-n)} + i\,\frac{\sin(\pi(1-n))}{\pi(1-n)^2}$$

$$= i\left(\frac{(-1)^n}{(1-n)} + 0\right) = i\,\frac{(-1)^n}{(1-n)}$$

$$\Rightarrow \sum c_n e^{inx} = \sum \frac{(-1)^n}{1-n}\, i e^{inx}$$

b $\operatorname{Im}\{f(x)\} = \operatorname{Im}\{x e^{ix}\}$

$$= \operatorname{Im}\{x(\cos x + i \sin x)\} = x \sin x$$

$$\operatorname{Im}\Big\{\sum_{n\in\mathbb{Z}} c_n e^{inx}\Big\} = \operatorname{Im}\Big\{\sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n}\, i e^{inx}\Big\}$$

$$= \operatorname{Im}\Big\{\sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n}\, i(\cos nx + i \sin nx)\Big\}$$

$$= \sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n} \cos nx$$

$$f(x) = \sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n}\, i e^{inx} \qquad -\pi < x < \pi$$

$$\operatorname{Im}\{f(x)\} = \operatorname{Im}\Big\{\sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n}\, i e^{inx}\Big\} \qquad -\pi < x < \pi$$

$$x \sin x = \sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n} \cos nx \qquad -\pi < x < \pi$$

c $n = 0$: $\frac{(-1)^n}{1-n} \cos nx = \frac{(-1)^0}{1-0} \cos(0\cdot x) = 1\cdot 1 = 1$

~~n = -1~~

$n = -1$: $\frac{(-1)^{-1}}{1-(-1)} \cos(-1\cdot x) = \frac{1}{2(-1)} \cos(-x) = -\frac{1}{2}\cos x$

$$x \sin x = \sum_{n\in\mathbb{Z}\setminus\{1\}} \frac{(-1)^n}{1-n} \cos nx \qquad -\pi < x < \pi$$

$$= 1 - \frac{1}{2}\cos x + \sum_{\substack{n\in\mathbb{Z}\\ |n|\geq 2}} \frac{(-1)^n}{1-n} \cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{(-1)^n}{1-n}\cos nx + \frac{(-1)^{-n}}{1-(-n)}\cos(-nx)\right)$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left(\frac{(-1)^n}{1-n}\cos nx + \frac{(-1)^n}{1+n}\cos nx\right)$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} (-1)^n \cos nx \left(\frac{1}{1-n} + \frac{1}{1+n}\right)$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} (-1)^n \cos nx \left(\frac{1+n+1-n}{1-n^2}\right)$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} (-1)^n \cos nx \frac{2}{1-n^2}$$

$$= 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx \qquad -\pi < x < \pi$$

d

-2 $x = \pi$ invullen bij c)

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4 a $(f*f)(x) = \int_{-1}^{1} f(x-t)\,dt$

$f(x-t) \neq 0$ voor $-1+x \le t \le x+1$

$\begin{pmatrix} 1-x \ge -t \ge -x-1 \\ 1 \ge x-t \ge -1 \end{pmatrix}$

- $x \ge 2 \Rightarrow 1 = -1+2 \le -1+x \le t \le 3 \le x+1$

$\Rightarrow (f*f)(x) = \int_{-1}^{1} f(x-t)\,dt = 0$

-1

- $0 \le x \le 2 \Rightarrow \quad 1 \le x+1 \le 3$

$-1 \le -1+x \le 1$

$(f*f)(x) = \int_{-1}^{1} \underbrace{f(x-t)}_{1 \text{ als } x-t \le 1,\ t \ge x-1} dt \qquad -t \le 1-x$

$= \int_{x-1}^{1} f(x-t)\,dt \qquad t-x \le 1,\ t \le 1+x,\ t \le 1 \le x+1$

$= \int_{x-1}^{1} 1\,dt$

$= [t]_{x-1}^{1} = 1-(x-1) = 2-x$

$= 2-|x|$, want $x \ge 0$

~~$x \le 0 \Rightarrow x+1 \le 1$~~

$-1+x \le -1$

$(f*f)(x) = \int_{-1}^{1} \underbrace{f(x-t)}_{=1 \text{ als } t-x \le 1 \Rightarrow t \le x+1 \le 1} dt$

$= \int_{-1}^{x+1} f(x-t)\,dt$

f even functie,

$= \int_{-1}^{x+1} dt = [t]_{-1}^{x+1} = x+1-(-1)$

$= 2+x = 2-|x|$, want $x \le 0$

$\vee\ -2 \le x \le 0$

dus $(f*f)(x) = \begin{cases} 2-|x| & |x| \le 2 \\ 0 & |x| > 2 \end{cases}$

b $\hat{f}(x) = \int_{\mathbb{R}} e^{ixt} f(t)\, dt$

$$= \int_{-1}^{1} e^{ixt}\, dt$$

$$= \left[\frac{e^{ixt}}{ix} \right]_{-1}^{1} = \frac{e^{ix} - e^{-ix}}{ix}$$

$$= \frac{2\sin x}{x}$$

c $\cancel{\hat{f} * \hat{f} = \hat{g}}$ $g = f * f$

$$\hat{g} = \widehat{f * f} = \hat{f}\,\hat{f} = \left(\frac{2\sin x}{x}\right)^2$$

d $\int_{\mathbb{R}} |\hat{g}(x)|^2\, dx = \int_{\mathbb{R}} \left| \frac{4\sin^2 x}{x^2} \right|^2 dx$

$$= \int_{\mathbb{R}} \frac{16 \sin^4 x}{x^4}\, dx = 16 \int_{\mathbb{R}} \frac{\sin^4 x}{x^4}\, dx$$

$$= \cancel{16}\ 2\pi \int_{\mathbb{R}} |g(t)|^2\, dt$$

$$= 2\pi \int_{-2}^{2} |2 - |t||^2\, dt$$

$$= 2\pi \left(\int_{-2}^{0} (2+t)^2\, dt + \int_{0}^{2} (2-t)^2\, dt \right)$$

$$= 2\pi \left(\int_{-2}^{0} (4 + 4t + t^2)\, dt + \int_{0}^{2} (4 - 4t + t^2)\, dt \right)$$

$$= 2\pi \left(\left[\underset{-8}{4t} + \underset{8}{2t^2} + \underset{-\frac{8}{3}}{\tfrac{1}{3}t^3} \right]_{-2}^{0} + \left[\underset{8}{4t} - \underset{-8}{2t^2} + \underset{\frac{8}{3}}{\tfrac{1}{3}t^3} \right]_{0}^{2} \right)$$

$$\cancel{= 2\pi \left(\left[4t - 2t^2 + \tfrac{1}{3}t^3 - 4t - 2t^2 - \tfrac{1}{3}t^3 \right]_{0}^{2} \right)}$$

$$\cancel{= 2\pi \left[-4t^2 \right]_{0}^{2} = 8\pi(-2^2 - 0) = -32\pi}$$

$$\cancel{\Rightarrow \int_{\mathbb{R}} \frac{\sin^4 x}{x^4}\, dx = -\frac{32\pi}{16} = -2\pi}$$

$$= 2\pi \left(\left(0 - \left(-8 + 8 - \tfrac{8}{3} \right) \right) + \left(\left(8 - 8 + \tfrac{8}{3} \right) - 0 \right) \right)$$

$$= 2\pi \left(-8 + 8 - \tfrac{8}{3} + 8 - 8 + \tfrac{8}{3} \right) = 0$$

$$\Rightarrow \int_{\mathbb{R}} \frac{\sin^4 x}{x^4}\, dx = 0$$

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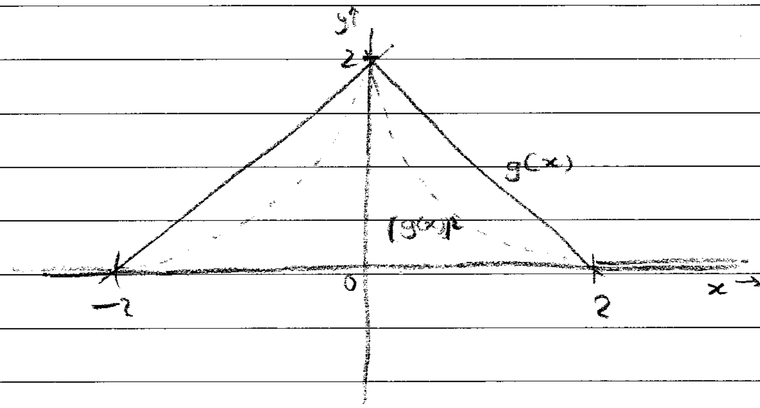

$$2\pi \int_{\mathbb{R}} |g(t)|^2 \, dt$$

$$= 4\pi \int_0^2 |g(t)|^2 \, dt$$

$$= 4\pi \int_0^2 (2-t)^2 \, dt$$

$$= 4\pi \int_0^2 (4 - 4t + t^2) \, dt$$

$$= 4\pi \left[4t - 2t^2 + \tfrac{1}{3}t^3 \right]_0^2$$

$$= 4\pi \left(8 - 8 + \tfrac{1}{3}\cdot 8 \right) = \frac{32\pi}{3}$$

$$16 \int_{\mathbb{R}} \frac{\sin^4 x}{x^4} \, dx = \frac{32\pi}{3}$$

$$\Rightarrow \int_{\mathbb{R}} \frac{\sin^4 x}{x^4} \, dx = \frac{2}{3}\pi$$